J. Math. Anal. Appl. 385 (2012) 683–692

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Traveling waves in a bio-reactor model with stage-structure

Zhi-Cheng Wang [a,b,*,1], Jianhua Wu [a,2]

[a] College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, Shaanxi 710062, China
[b] School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, China

ARTICLE INFO

Article history:
Received 31 March 2010
Available online 6 July 2011
Submitted by M. Iannelli

Keywords:
Bio-reactor model
Traveling wave solutions
Stage-structure
Fixed point theorem

ABSTRACT

In this paper the existence and non-existence of traveling wave solutions are established for a bio-reactor model with stage-structure. The method to prove the existence of wave solutions is to construct an invariant cone of initial functions defined in a large but bounded domain, to apply a fixed point theorem on this cone and then extend to the unbounded spatial domain $\mathbb{R}$ by a limiting argument.

1. Introduction

In [1], the authors derived an autonomous parabolic system to model microbial growth and competition in a flow reactor:

$$\begin{cases} S_t = d_1 S_{xx} - v S_x - f(S)u, \\ u_t = u_{xx} - v u_x + f(S)u - ku, \end{cases} \tag{1.1}$$

where $S(x,t)$ and $u(x,t)$ are the concentrations of nutrient and microbial population at position x and time t, respectively. Parameter $v \geqslant 0$ is the flow velocity, $k > 0$ is the cell death rate, and $d_1 > 0$ is the ratio of the diffusivity of the nutrient to the random cell motility coefficient of the organism. The periodic case is treated in [4]. When $f(S) = \beta S$, $d_1 = 0$ and $v = 0$, (1.1) was derived for the spatial spread of rabies in Europe carried by red foxes [12,13].

Kennedy and Aris [3] showed the existence of traveling wave solutions of (1.1) with $v = 0$ and $d_1 = 0$ and conjectured the existence of such waves when $d_1 > 0$. Smith and Zhao [18] gave a proof of this conjecture for small positive d_1 by applying the geometric singular perturbation theory and the center manifold theorem, where $f' > 0$ is required. When $f(S) = \beta S$, (1.1) was studied as an epidemic model by Hosono and Ilyas [10] who showed that (1.1) admits traveling wave solutions for any $d_1 > 0$. Furthermore, Huang [11] gave a complete proof of the conjecture and established the existence of traveling wave solutions of (1.1) for any diffusion coefficient $d_1 > 0$ and more general function f. His approach is a shooting method with the aid of an appropriately constructed Liapunov function.

However, a delay in nutrient recycling to viable biomass is always present in nature [2,8,21]. Let the constant $\tau > 0$ denote the time delay involved in the conversion of nutrient to viable biomass, $D \geqslant 0$ denote the average diffusion coefficient of the intermediate state which is between 0 (diffusion coefficient for the nutrient) and 1 (diffusion coefficient for

* Corresponding author at: School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, China.
E-mail address: wangzhch@lzu.edu.cn (Z.-C. Wang).

1 This work was partially supported by the National Natural Science Foundation of China (11071105), by Program for New Century Excellent Talents in University (NCET-10-0470) and by the China Postdoctoral Science Foundation (20100481319).
2 Supported by the Natural Science Foundation of China (No. 10971124).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.06.084

the microbial biomass), and the parameter $d \geqslant 0$ denote the dilution rate of the intermediate state. Then using a similar argument to [14,15], we can derive the following model

$$\begin{cases} S_t = d_1 S_{xx} - vS_x - f(S)u, \\ u_t = u_{xx} - vu_x - ku + e^{-d\tau} \displaystyle\int_{-\infty}^{+\infty} f\big(S(y - v\tau, t - \tau)\big)u\big(y - v\tau, t - \tau\big) f_\alpha(x - y)\,dy, \end{cases} \tag{1.2}$$

where $d_1 \geqslant 0$, $k > 0$, $f_\alpha(x) = \sqrt{4\pi\alpha}e^{-\frac{x^2}{4\alpha}}$, $\alpha = \tau D$. When $f(S) = \beta S$ and $v = 0$, system (1.2) was exactly derived by Li and Zou [14] to model spatial spread of infectious diseases. In fact, Li and Zou [14] derived a more general model. When $d_1 = 0$, Ou and Wu [17] showed that system (1.2) admits nontrivial traveling wave solutions for sufficiently small $\tau > 0$, see also Yang et al. [22].

In this paper, we consider the traveling wave solutions of (1.2). Throughout this paper we always assume that the function f satisfies the following assumptions:

(A1) $f \in C^2(\mathbb{R})$, $f(0) = 0$, $f'(s) \geqslant 0$ for $s > 0$;
(A2) There exists a unique $S^* > 0$ such that $e^{-d\tau} f(S^*) = k$, $e^{-d\tau} f(S) < k$ for $S \in (0, S^*)$ and $e^{-d\tau} f(S) > k$ for $S > S^*$;
(A3) For every $S^0 > 0$, there exists $K_0 > 0$ such that

$$\left| \frac{f(x)}{x} - \frac{f(y)}{y} \right| \leqslant K_0|x - y| \quad \text{for } x, y \in \big(0, S^0\big).$$

Whenever $d_1 \geqslant 0$ and $\tau > 0$, we show that for $S^0 > S^*$ there exists $c^* > 0$ such that system (1.2) admits a traveling wave solution $(S(x + ct), u(x + ct))$ with $c > c^*$ satisfying

$$S(-\infty) = S^0 > S_0 = S(+\infty) \quad \text{and} \quad u(\pm\infty) = 0. \tag{1.3}$$

Our method is to construct an appropriately invariant cone of initial functions defined in a large but bounded domain, apply a fixed point theorem on this cone for the relevant solution operators and then extend to the unbounded spatial domain $\mathbb{R}$ by a limiting argument. This method is motivated by [5] and [6], where the existence and non-existence of traveling wave solutions for some infection-age structured epidemic models with diffusion are studied. It was also successfully applied to establish the existence and non-existence of traveling wave solutions of a reaction–convection epidemic model for the spatial spread of H5N1 avian influenza involving a wide range of bird species and environmental contamination [20,16]. Consequently, we show that there is no traveling wave solutions of (1.2) satisfying (1.3) if $S^0 < S^*$ whenever $d_1 \geqslant 0$ and $\tau > 0$. Thus, we establish a threshold condition for the existence of nontrivial traveling wave solutions of (1.2).

As mentioned above, system (1.2) is usually regarded as an epidemic model when $f(S) = \beta S$. In this case $\tau > 0$ represents the latent period and $d > 0$ represents the removed rate in the latent period which includes the disease mortality rate, the recover rate and the natural death rate, see [14]. Let $R_0 = \frac{\beta S^0}{ke^{d\tau}}$, which is usually called the basic reproduction number. Obviously, the above results imply that the existence and non-existence of nontrivial traveling wave solutions are completely determined by the basic reproduction number R_0. An important character of such a basic reproduction number is involving the latent period and the removed rate in the latent period.

2. Existence of nontrivial traveling wave solutions

In this section, we give an exactly proof for the existence of traveling wave solutions of (1.2) when $S^0 > S^*$. In the sequel, we only treat the case $d_1 > 0$.

For $S^0 > S^*$, we look for a traveling wave solution of (1.2), that is, a solution with the form $S(x, t) = S(x + ct)$ and $u(x, t) = u(x + ct)$, which satisfies the following equations

$$cS' = d_1 S'' - vS' - f(S)u, \tag{2.1}$$

$$cu' = u'' - vu' - ku + e^{-d\tau} \int_{-\infty}^{+\infty} f\big(S\big(\xi - y - (c + v)\tau\big)\big)u\big(\xi - y - (c + v)\tau\big) f_\alpha(y)\,dy, \tag{2.2}$$

and the following boundary conditions

$$u(\pm\infty) = 0, \qquad S(-\infty) = S^0 \quad \text{and} \quad S(+\infty) = S_0 < S^0.$$

Linearizing (2.2) at $(S, u) \equiv (S^0, 0)$, we have a linearized equation

$$c\varphi' = \varphi'' - v\varphi' + e^{-d\tau} \int_{-\infty}^{+\infty} f\big(S^0\big)\varphi\big(\xi - y - (c + v)\tau\big) f_\alpha(y)\,dy - k\varphi.$$

Letting $\varphi(\xi) = e^{\lambda\xi}$ yields the characteristic equation

$$\Theta(\lambda, c) := \lambda^2 - (c+v)\lambda + e^{-d\tau} f\big(S^0\big) e^{\alpha\lambda^2 - \tau(c+v)\lambda} - k = 0.$$

Since $e^{-d\tau} f(S^0) > k$, then there exist $c_* > 0$ and $\lambda_* > 0$ such that (i) $\Theta(\lambda_*, c_* - v) = 0$, $|\frac{\partial}{\partial\lambda}\Theta(\lambda, c)|_{(\lambda_*, c_*-v)} = 0$; (ii) when $c > c_* - v$, there exist $\lambda_1(c) > 0$ and $\lambda_2(c) > 0$ satisfying (a) $0 < \lambda_1(c) < \lambda_* < \lambda_2(c)$; (b) $\Theta(\lambda, c) > 0$ for $\lambda \in (-\infty, \lambda_1(c)) \cup (\lambda_2(c), +\infty)$ and $\Theta(\lambda, c) < 0$ for $\lambda \in (\lambda_1(c), \lambda_2(c))$.

Let $c > c_* - v$ and denote $\lambda_1(c)$ by λ_1. Define

$$\phi_-(\xi) = \max\big\{S^0\big(1 - \beta e^{\mu\xi}\big), 0\big\}, \qquad \psi_+(\xi) = e^{\lambda_1\xi} \quad \text{and} \quad \psi_-(\xi) = \max\big\{e^{\lambda_1\xi}\big(1 - Me^{\epsilon\xi}\big), 0\big\},$$

where $\beta > 0$, $\mu > 0$, $\epsilon > 0$, $M > 0$ will be determined in the sequel.

Lemma 2.1. *$\psi_+(\xi) = e^{\lambda_1\xi}$ satisfies the following equation*

$$c\psi_+'(\xi) = \psi_+''(\xi) - v\psi_+'(\xi) + e^{-d\tau} \int_{-\infty}^{+\infty} f\big(S^0\big)\psi_+\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\, dy - k\psi_+(\xi). \tag{2.3}$$

Lemma 2.2. *Let $\mu < \frac{\lambda_1}{2}$ and $\beta > 0$ large enough. Then $\phi_-(\xi) = \max\{S^0(1 - \beta e^{\mu\xi}), 0\}$ satisfies the following inequality*

$$c\phi_-' \leqslant d_1\phi_-'' - v\phi_-' - g_{\max}\phi_-(\xi)\psi_+(\xi) \tag{2.4}$$

for $\xi < -\frac{1}{\mu}\ln\beta$, where $g_{\max} = \sup_{\xi\in(0,S^0]}\{f(\xi)/\xi\}$.

Proof. When $x < -\frac{1}{\mu}\ln\beta$, $\phi_-(\xi) = S^0(1 - \beta e^{\mu\xi})$. Then we have

$$\begin{aligned}
(c+v)\phi_-'(\xi) - d_1\phi_-''(\xi) + g_{\max}\phi_-(\xi)\psi_+(\xi) &= -(c+v)\mu\beta S^0 e^{\mu\xi} + \mu^2 d_1\beta S^0 e^{\mu\xi} + g_{\max}\phi_-(\xi)e^{\lambda_1\xi} \\
&\leqslant -(c+v)\mu\beta S^0 e^{\mu\xi} + \mu^2 d_1\beta S^0 e^{\mu\xi} + g_{\max} S^0 e^{\lambda_1\xi} \\
&\leqslant \big[-(c+v)\mu\beta S^0 + \mu^2 d_1\beta S^0 + g_{\max} S^0 e^{(\lambda_1-\mu)\frac{1}{\mu}\ln\frac{1}{\beta}}\big] e^{\mu\xi} \\
&= \big[-(c+v)\mu\beta S^0 + \mu^2 d_1\beta S^0 + g_{\max} S^0 \beta^{-\frac{\lambda_1-\mu}{\mu}}\big] e^{\mu\xi} \\
&\leqslant \big[-(c+v)\mu\beta S^0 + \mu^2 d_1\beta S^0 + g_{\max} S^0 \beta^{-1}\big] e^{\mu\xi} \\
&\leqslant 0
\end{aligned}$$

if $\mu < \frac{c+v}{2d_1}$ and $\beta > \sqrt{\frac{2g_{\max}S^0}{(c+v)\mu S^0}}$. This completes the proof. $\square$

Lemma 2.3. *Let $\epsilon < \min\{\frac{\mu}{2}, \lambda_2 - \lambda_1\}$. Then $\psi_-(\xi) = \max\{e^{\lambda_1\xi}(1 - Me^{\epsilon\xi}), 0\}$ satisfies the following inequality*

$$-\psi_-''(\xi) + (c+v)\psi_-'(\xi) + k\psi_-(\xi) \leqslant e^{-d\tau} \int_{-\infty}^{+\infty} f\big(\phi_-\big(\xi - y - (c+v)\tau\big)\big)\psi_-\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\, dy \tag{2.5}$$

for $\xi < -\frac{1}{\epsilon}\ln M$, where $M > 0$ is sufficiently large so that $-\frac{1}{\epsilon}\ln M < -\frac{1}{\mu}\ln\beta$.

Proof. Let $\overline{f}_0 = \max_{u\in[0,S^0]} f'(u)$. Note that $\Delta(\lambda_1 + \epsilon, c) < 0$. When $x < -\frac{1}{\epsilon}\ln M$, $\psi_-(\xi) = e^{\lambda_1\xi}(1 - Me^{\epsilon\xi}) > 0$. Then one has

$$\begin{aligned}
&-\psi_-''(\xi) + (c+v)\psi_-'(\xi) + k\psi_-(\xi) - e^{-d\tau} \int_{-\infty}^{+\infty} f\big(\phi_-\big(\xi - y - (c+v)\tau\big)\big)\psi_-\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\, dy \\
&\quad\leqslant -\psi_-''(\xi) + (c+v)\psi_-'(\xi) + k\psi_-(\xi) - e^{-d\tau} f\big(S^0\big) \int_{-\infty}^{+\infty} \psi_-\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\, dy \\
&\qquad + e^{-d\tau}\overline{f}_0 \int_{-\infty}^{+\infty} \big(S^0 - \phi_-\big(\xi - y - (c+v)\tau\big)\big)\psi_-\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\, dy
\end{aligned}$$

$$
\begin{aligned}
&\leqslant -\Delta(\lambda_1, c)e^{\lambda_1\xi} + \Delta(\lambda_1+\epsilon, c)Me^{(\lambda_1+\epsilon)\xi} + e^{-d\tau}\overline{f}_0\int_{-\infty}^{+\infty} S^0\beta e^{\mu(\xi-y-(c+v)\tau)}e^{\lambda_1(\xi-y-(c+v)\tau)}f_\alpha(y)\,dy \\
&= \Delta(\lambda_1+\epsilon, c)Me^{(\lambda_1+\epsilon)\xi} + e^{-d\tau}\overline{f}_0 S^0\beta e^{-(\lambda_1+\mu)(c+v)\tau}e^{-\alpha(\mu+\lambda_1)^2}e^{(\lambda_1+\mu)\xi} \\
&\leqslant \big[\Delta(\lambda_1+\epsilon, c)M + e^{-d\tau}\overline{f}_0 S^0\beta e^{-(\mu-\epsilon)\frac{1}{\epsilon}\ln M}\big]e^{(\lambda_1+\epsilon)\xi} \\
&\leqslant \left[\Delta(\lambda_1+\epsilon, c)M + \frac{1}{M}e^{-d\tau}\overline{f}_0 S^0\beta\right]e^{(\lambda_1+\epsilon)\xi} \\
&< 0
\end{aligned}
$$

for large enough $M > 0$. This completes the proof. $\square$

Now we establish the existence of traveling wave solutions of (1.2) for $S^0 > S^*$. Take $X > \frac{1}{\epsilon}\ln M$. Let

$$
\Gamma_X = \left\{ \big(\chi(\cdot), \varphi(\cdot)\big) \in C\big([-X, X], \mathbb{R}^2\big) \;\middle|\; \begin{array}{l} \chi(\pm X) = \phi_-(\pm X),\ \varphi(\pm X) = \psi_-(\pm X), \\ \phi_-(\xi) \leqslant \chi(\xi) \leqslant S^0,\ \psi_-(\xi) \leqslant \varphi(\xi) \leqslant \psi_+(\xi), \\ \text{for any } \xi \in [-X, X] \end{array} \right\}.
$$

Now for any $(\chi(\cdot), \varphi(\cdot)) \in \Gamma_X$, define

$$
\hat{\chi}(\xi) = \begin{cases} \chi(\xi), & |\xi| \leqslant X, \\ \phi_-(\xi), & |\xi| > X, \end{cases} \qquad \hat{\varphi}(\xi) = \begin{cases} \varphi(\xi), & |\xi| \leqslant X, \\ \psi_-(\xi), & |\xi| > X. \end{cases}
$$

Consider the following two-point boundary value problem:

$$
-d_1 S''(\xi) + (c+v)S'(\xi) + g[\chi](\xi)\varphi(\xi)S(\xi) = 0, \tag{2.6}
$$

$$
-u''(\xi) + (c+v)u'(\xi) + ku(\xi) = e^{-d\tau}\int_{-\infty}^{+\infty} f\big(\hat{\chi}\big(\xi-y-(c+v)\tau\big)\big)\hat{\varphi}\big(\xi-y-(c+v)\tau\big)f_\alpha(y)\,dy, \tag{2.7}
$$

with

$$
S(\pm X) = \phi_-(\pm X) \quad \text{and} \quad u(\pm X) = \psi_-(\pm X), \tag{2.8}
$$

where

$$
g[\chi](x) = \begin{cases} f(\chi(x))/\chi(x), & \chi(x) > 0, \\ f'(0), & \chi(x) = 0. \end{cases}
$$

It is obvious that $g[\chi](x)$ is continuous in $x \in [-X, X]$. By the ODE theory and the elliptic PDE theory [7] we have that the problem (2.6)–(2.8) admits a unique solution $(S_X(\cdot), u_X(\cdot))$ satisfying $S_X(\cdot) \in W^{2,p}(-X, X)$ and $u_X(\cdot) \in C^2[-X, X]$, where $p \geqslant 2$. In particular, $S_X(\cdot)$ is a strong solution of (2.6), that is, $S_X(\cdot)$ satisfies (2.6) on a.e. $[-X, X]$. By the embedding theorem, we have $S_X(\cdot) \in W^{2,p}(-X, X) \hookrightarrow C^{1+\alpha}[-X, X]$ for some $\alpha \in (0, 1)$. Consequently, we define an operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) : \Gamma_X \to C[-X, X]$ by

$$
S_X(\xi) = \mathcal{A}_1[\chi, \varphi](\xi) \quad \text{and} \quad u_X(\xi) = \mathcal{A}_2[\chi, \varphi](\xi) \quad \text{for } \xi \in [-X, X]. \tag{2.9}
$$

Theorem 2.4. *The operator $\mathcal{A}$ maps Γ_X into Γ_X.*

Proof. Since 0 is a subsolution of (2.6), by the maximum principle we have $S_X(\xi) \geqslant 0$ for $\xi \in [-X, X]$. Similarly, we have $S_X(\xi) \leqslant S^0$ for $\xi \in [-X, X]$. Note that $\phi_-(\xi) = S^0(1 - \beta e^{\mu\xi})$ for $\xi \in [-X, X']$ with $X' = -\frac{1}{\mu}\ln\beta$ and satisfies (2.4). It follows that

$$
\begin{aligned}
0 &\geqslant -d_1\phi_-'' + (c+v)\phi_-' + g_{\max}\psi_+(\xi)\phi_-(\xi) \\
&\geqslant -d_1\phi_-'' + (c+v)\phi_-' + \frac{f(\chi(\xi))}{\chi(\xi)}\varphi(\xi)\phi_-(\xi) \\
&= -d_1\phi_-'' + (c+v)\phi_-' + g\big(\chi(\xi)\big)\varphi(\xi)\phi_-(\xi)
\end{aligned}
$$

for any $\xi \in (-X, X')$. Since $S_X(-X) = \phi_-(-X)$ and $S_X(X') \geqslant \phi_-(X') = 0$, the maximum principle implies that $\phi_-(\xi) \leqslant S_X(\xi)$ for $\xi \in [-X, X']$. Hence, $\phi_-(\xi) \leqslant S_X(\xi) \leqslant S^0$ for $\xi \in [-X, X]$.

Consider $u_X(\xi)$. Firstly, we have $u_X(\xi) \geqslant 0$ for $\xi \in [-X, X]$. Since $\phi_-(\xi) \leqslant \hat{\chi}(\xi) \leqslant S^0$ and $\psi_-(\xi) \leqslant \hat{\varphi}(\xi) \leqslant \psi_+(\xi)$ for $\xi \in \mathbb{R}$, then we have

$$
\begin{aligned}
& e^{-d\tau} \int_{-\infty}^{+\infty} f\big(\phi_-\big(\xi - y - (c+v)\tau\big)\big)\psi_-\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\,dy \\
& \quad \leqslant e^{-d\tau} \int_{-\infty}^{+\infty} f\big(\hat{\chi}\big(\xi - y - (c+v)\tau\big)\big)\hat{\varphi}\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\,dy \\
& \quad \leqslant e^{-d\tau} \int_{-\infty}^{+\infty} f\big(S^0\big)\psi_+\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\,dy
\end{aligned}
$$

for $\xi \in \mathbb{R}$, which combining (2.3) and (2.5) imply that $\psi_-(\xi)$ and $\psi_+(\xi)$ are subsolution and supersolution of (2.7) on $(-X, X'')$ and $(-X, X)$ with $X'' = -\frac{1}{\epsilon}\ln M$, respectively. Therefore, we further have $\psi_-(\xi) \leqslant u_X(\xi) \leqslant \psi_+(\xi)$ for $\xi \in [-X, X]$. This completes the proof. □

Theorem 2.5. *The operator $\mathcal{A} : \Gamma_X \to \Gamma_X$ is completely continuous.*

Proof. Firstly, the standard elliptic estimates and the embedding theorem imply that $\mathcal{A}$ is compact. Next, we prove that $\mathcal{A}$ is continuous. It is obvious that $\mathcal{A}_2$ is continuous. Consider $\mathcal{A}_1$. Assume that $(\chi_i, \varphi_i) \in \Gamma_X$, $i = 1, 2$, and $S_{X,i}(x) = \mathcal{A}_1[\chi_i, \varphi_i](\xi)$ for $\xi \in [-X, X]$. Then we have

$$
\begin{aligned}
& -d_1[S_{X,1} - S_{X,2}]'' + (c+v)[S_{X,1} - S_{X,2}]' + g[\chi_1](\xi)\varphi_1(\xi)\big[S_{X,1}(\xi) - S_{X,2}(\xi)\big] \\
& \quad = \big[g[\chi_2](\xi)\varphi_2(\xi) - g[\chi_1](\xi)\varphi_1(\xi)\big]S_{X,2}(\xi) \\
& \quad = \big[g[\chi_2](\xi) - g[\chi_1](\xi)\big]\varphi_1(\xi)S_{X,2}(\xi) + g[\chi_2](\xi)S_{X,2}(\xi)\big[\varphi_2(\xi) - \varphi_1(\xi)\big]
\end{aligned}
$$

for $\xi \in (-X, X)$. Again applying the standard elliptic estimates and the embedding theorem, it is sufficient to prove that $g[\chi](\cdot)$ is continuous on the function $\chi \in C[-X, X]$ with $\phi_-(\xi) \leqslant \chi(\xi) \leqslant S^0$ for $\xi \in [-X, X]$. When $\chi_1(\xi)\chi_2(\xi) > 0$, we have

$$
\big|g[\chi_2](\xi) - g[\chi_1](\xi)\big| = \left|\frac{f(\chi_2(\xi))}{\chi_2(\xi)} - \frac{f(\chi_1(\xi))}{\chi_1(\xi)}\right| \leqslant K_0\|\chi_1 - \chi_2\|_{C[-X,X]}
$$

due to the assumption (A3). When $\chi_2(\xi) > 0$ and $\chi_1(\xi) = 0$, we have

$$
\big|g[\chi_2](\xi) - g[\chi_1](\xi)\big| = \left|\frac{f(\chi_2(\xi))}{\chi_2(\xi)} - f'(0)\right| \leqslant K_0\|\chi_1 - \chi_2\|_{C[-X,X]}.
$$

Similarly, we have $|g[\chi_2](\xi) - g[\chi_1](\xi)| \leqslant K_0\|\chi_1 - \chi_2\|_{C[-X,X]}$ for $\chi_1(\xi) \geqslant 0$ and $\chi_2(\xi) = 0$. Thus, $g[\chi](\cdot)$ is continuous on the function χ. This completes the proof. □

Furthermore, it is easy to see that Γ_X is closed and convex. Consequently, the Schauder's fixed point theorem implies that there exists $(S_X, u_X) \in \Gamma_X$ such that

$$
(S_X, u_X)(\xi) = \mathcal{A}[S_X, u_X](\xi)
$$

for any $\xi \in [-X, X]$. The following theorem lists some estimates about S_X and u_X.

Theorem 2.6. *There exists a constant $C_0 > 0$ such that*

$$
\|S_X\|_{C^3[-X,X]} < C_0, \qquad \|u_X\|_{C^{2,1}[-X,X]} < C_0 \quad \textit{and} \quad \int_{-X}^{X} u_X(\xi)\,d\xi \leqslant C_0
$$

for any $X > \frac{1}{\epsilon}\ln M$. In particular, $S'_X(\xi) \leqslant 0$.

Proof. Firstly, we have that (S_X, u_X) satisfy

$$
-d_1 S''_X + (c+v)S'_X + g[S_X](\xi)u_X(\xi)S_X(\xi) = 0, \quad \text{a.e. } \xi \in (-X, X)
$$

and

$$-u_X''(\xi)+(c+v)u_X'(\xi)+ku_X(\xi)=e^{-d\tau}\int_{-\infty}^{+\infty} f\big(\widehat{S}_X\big(\xi-y-(c+v)\tau\big)\big)\hat{u}_X\big(\xi-y-(c+v)\tau\big)f_\alpha(y)\,dy \tag{2.10}$$

for any $\xi\in(-X,X)$, where

$$\widehat{S}_X(\xi)=\begin{cases} S_X(\xi), & |\xi|\leqslant X,\\ \phi_-(\xi), & |\xi|>X,\end{cases}\qquad \hat{u}_X(\xi)=\begin{cases} u_X(\xi), & |\xi|\leqslant X,\\ \psi_-(\xi), & |\xi|>X.\end{cases}$$

By the embedding theorem, we have that $S_X\in W^{2,p}(-X,X)\hookrightarrow C^{1,\alpha}[-X,X]$ for some $\alpha\in(0,1)$. Noting that $u_X\in C^2[-X,X]$, we then have $S_X\in C^3[-X,X]$ and $u_X\in C^{2,1}[-X,X]$. Hence, S_X satisfies

$$-d_1S_X''+(c+v)S_X'+f\big(S_X(\xi)\big)u_X(\xi)=0,\quad \forall\xi\in[-X,X]. \tag{2.11}$$

Following the last equality, we have

$$\big(S_X'(\xi)e^{-\frac{c+v}{d_1}\xi}\big)'=e^{-\frac{c+v}{d_1}\xi}f\big(S_X(\xi)\big)u_X(\xi),\quad \forall\xi\in[-X,X].$$

Integrating the last equality from $\xi\in[-X,X)$ to X, we have

$$S_X'(\xi)=S_X'(X)e^{-\frac{c+v}{d_1}(X-\xi)}-\int_{\xi}^{X} e^{-\frac{c+v}{d_1}(x-\xi)}f\big(S_X(x)\big)u_X(x)\,dx.$$

Since $S_X(\xi)\geqslant 0=S_X(X)$ for $\xi\in[-X,X]$, we have $S_X'(X)\leqslant 0$ and hence, $S_X'(\xi)\leqslant 0$ for $\xi\in[-X,X]$. In particular, $S_X'(\xi)\not\equiv 0$.

Integrating (2.11) on $[-X,X]$, we have

$$\begin{aligned}\int_{-X}^{X} f\big(S_X(\xi)\big)u_X(\xi)\,d\xi &= (c+v)\big(S_X(-X)-S_X(X)\big)+d_1\big(S_X'(X)-S_X'(-X)\big)\\ &\leqslant (c+v)S^0-d_1\phi_-'(-X)\end{aligned}$$

due to $S_X(-X)\leqslant S^0$ and $S_X'(-X)\geqslant\phi_-'(-X)$. Noting that the definition of $\widehat{S}_X$ and $\hat{u}_X$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f\big(\widehat{S}_X(\xi)\big)\hat{u}_X(\xi)\,d\xi &= \int_{-\infty}^{-X} f\big(\phi_-(\xi)\big)\psi_-(\xi)\,d\xi+\int_{-X}^{X} f\big(S_X(\xi)\big)u_X(\xi)\,d\xi\\ &\leqslant \int_{-\infty}^{-\frac{1}{\epsilon}\ln M} f\big(\phi_-(\xi)\big)\psi_-(\xi)\,d\xi+(c+v)S^0-d_1\phi_-'(-X).\end{aligned}$$

Now integrating the two sides of (2.10) on $[-X,X]$, we obtain

$$\begin{aligned}\int_{-X}^{X} u_X(\xi)\,d\xi &= \frac{1}{k}\big[u_X'(X)-u_X'(-X)\big]+\frac{c+v}{k}\big[u_X(-X)-u_X(X)\big]\\ &\quad+\frac{e^{-d\tau}}{k}\int_{-X}^{+X}\int_{-\infty}^{+\infty} f\big(\widehat{S}_X\big(\xi-y-(c+v)\tau\big)\big)\hat{u}_X\big(\xi-y-(c+v)\tau\big)f_\alpha(y)\,dy\,d\xi\\ &\leqslant \frac{c+v}{k}\psi_-(-X)+\frac{e^{-d\tau}}{k}\left[\int_{-\infty}^{-\frac{1}{\epsilon}\ln M} f\big(\phi_-(\xi)\big)\psi_-(\xi)\,d\xi+(c+v)S^0-d_1\phi_-'(-X)\right].\end{aligned}$$

In the last inequality, we have used the facts that $u_X'(X)\leqslant 0$, $u_X'(-X)\geqslant\psi_-'(-X)>0$, $u_X(-X)=\psi_-(-X)$ and $\int_{-\infty}^{\infty} f_\alpha(y)\,dy=1$. Thus, we have that there exists $C_0>0$ independent of $X>\frac{1}{\epsilon}\ln M$ such that

$$\int_{-X}^{X} f\big(S_X(\xi)\big)u_X(\xi)\,d\xi\leqslant C_0,\qquad \int_{-\infty}^{\infty} f\big(\widehat{S}_X(\xi)\big)\hat{u}_X(\xi)\,d\xi\leqslant C_0,\qquad \int_{-X}^{X} u_X(\xi)\,d\xi\leqslant C_0.$$

Since $u_X'(-X) > 0$, there exists $\xi_0 \in (-X, X)$ such that $u_X(\xi_0) = \max_{\xi\in[-X,X]} u_X(\xi)$. Consequently, integrating both sides of (2.10) from $-X$ to ξ_0, we have

$$\begin{aligned} u_X(\xi_0) &= -\frac{1}{c+v}u_X'(-X) + \psi_-(-X) - \frac{k}{c+v}\int_{-X}^{\xi_0} u_X(\xi)\,d\xi \\ &\quad + \frac{e^{-d\tau}}{k}\int_{-X}^{\xi_0}\int_{-\infty}^{+\infty} f\big(\widehat{S}_X\big(\xi - y - (c+v)\tau\big)\big) \times \hat{u}_X\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\,dy\,d\xi \\ &\leqslant C_0 \end{aligned}$$

for some $C_0 > 0$ independent of $X > \frac{1}{\epsilon}\ln M$.

Integrating (2.11) from $-X$ to $\xi \in [-X, X]$, we obtain

$$\begin{aligned} -S_X'(\xi) &= -S_X'(-X) + \frac{c+v}{d_1}\big[S_X(-X) - S_X(\xi)\big] - \int_{-X}^{\xi} f\big(S_X(x)\big)u_X(x)\,dx \\ &\leqslant -\psi_-'(-X) + \frac{c+v}{d_1}S^0. \end{aligned}$$

Since

$$\big|S_X''(\xi)\big| \leqslant \frac{1}{d_1}\big[\big|S_X'(\xi)\big| + f\big(S^0\big)u_X(\xi_0)\big],$$

there exists some $C_0 > 0$ independent of $X > \frac{1}{\epsilon}\ln M$ such that $\|S_X\|_{C^2[-X,X]} < C_0$. Similarly, integrating (2.10) from $-X$ to $\xi \in [-X, X]$ and from $\xi \in [-X, X]$ to X, respectively, we can get $\|u_X\|_{C^2[-X,X]} < C_0$ for some $C_0 > 0$ independent of $X > \frac{1}{\epsilon}\ln M$.

Furthermore, differentiating both sides of (2.11) yields $\|S_X\|_{C^3[-X,X]} < C_0$ for some $C_0 > 0$ independent of $X > \frac{1}{\epsilon}\ln M$. By the definitions of $\widehat{S}_X$ and $\hat{u}_X$, we have that $\widehat{S}_X$ and $\hat{u}_X$ are uniformly Lipschitz continuous on $(-\infty, \infty)$ with a Lipschitz coefficient independent of $X > \frac{1}{\epsilon}\ln M$. Then, it follows from (2.10) that $\|u_X\|_{C^{2,1}[-X,X]} < C_0$ for some $C_0 > 0$ independent of $X > \frac{1}{\epsilon}\ln M$. This completes the proof. □

Now we state our main results in this paper. Let $\{X_n\}$ be an increasing sequence such that $X_n > \frac{1}{\epsilon}\ln M$ and $\lim_{n\to\infty} X_n = +\infty$. Then the solutions

$$(S_{X_n}, u_{X_n}) \in \Gamma_{X_n}$$

satisfy Theorem 2.6, (2.10) and (2.11). From the sequence

$$(S_{X_n}, u_{X_n}) \in \Gamma_{X_n},$$

we can extract a subsequence, still denoted by (S_{X_n}, u_{X_n}), tending towards functions $(S_*, u_*) \in C^2(\mathbb{R})$ in the following topologies

$$S_{X_n} \to S_* \quad \text{and} \quad u_{X_n} \to u_* \quad \text{in } C^2_{loc}(\mathbb{R}).$$

Applying the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} &\int_{-\infty}^{+\infty} f\big(\widehat{S}_{X_n}\big(\xi - y - (c+v)\tau\big)\big)\hat{u}_{X_n}\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\,dy \\ &\quad \to \int_{-\infty}^{+\infty} f\big(S_*\big(\xi - y - (c+v)\tau\big)\big)u_*\big(\xi - y - (c+v)\tau\big) f_\alpha(y)\,dy \end{aligned}$$

for any $\xi \in \mathbb{R}$ as $n \to +\infty$. Furthermore, (S_*, u_*) satisfy Eqs. (2.1) and (2.2), and

$$\int_{-\infty}^{\infty} u_*(x)\,dx < C_0, \qquad \phi_-(x) \leqslant S_*(x) \leqslant S^0, \qquad \psi_-(x) \leqslant u_*(x) \leqslant \min\big\{C_0, \psi_+(x)\big\}$$

for $x \in \mathbb{R}$. In addition, since $S_{X_n}'(x) \leqslant 0$, we have $S_*'(x) \leqslant 0$ for $x \in \mathbb{R}$.

Let $S_*(+\infty) = S_0$. Then there must be $S_0 < S^0$. In fact, if $S_0 = S^0$, it follows that $S_*(x) = S^0$ for all $x \in \mathbb{R}$. Consequently, $u_*(x) \equiv 0$, but it is impossible. Furthermore, applying the fluctuation lemma [9] we can show that $S'_*(\pm\infty) = 0$. Hence, we have

$$\int_{-\infty}^{+\infty} f\big(S_*(x)\big)u_*(x)\,dx = (c+v)\big(S^0 - S_0\big) > 0$$

and

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f\big(S_*\big(x-y-(c+v)\tau\big)\big)u_*\big(x-y-(c+v)\tau\big)f_\alpha(y)\,dy\,dx = (c+v)\big(S^0 - S_0\big) > 0.$$

It follows from $\int_{-\infty}^{\infty} u_*(x)\,dx < C_0$ that $u_*(\pm\infty) = 0$ and $u'_*(\pm\infty) = 0$. Consequently, we have

$$\int_{-\infty}^{\infty} u_*(x)\,dx = \frac{1}{k}e^{-d\tau}(c+v)\big(S^0 - S_0\big) \quad \text{and} \quad u_*(x) \leqslant e^{-d\tau}\big(S^0 - S_0\big) \tag{2.12}$$

for $x \in \mathbb{R}$. Now we have the following theorem.

Theorem 2.7. *Assume that $S^0 > S^*$. Then for any $c > c_*$, systems* (2.1) *and* (2.2) *admit a nonnegative traveling wave solution $(S_*(\xi), u_*(\xi))$ with $\xi = x + ct$ satisfying $S'_*(\xi) \leqslant 0$, $S_*(-\infty) = S^0$, $S_*(+\infty) = S_0 < S^0$, $u_*(\pm\infty) = 0$ and* (2.12).

3. Nonexistence of nontrivial traveling wave solutions

In this section we present the non-existence of nontrivial traveling wave solutions of (2.1) and (2.2) when $S^0 < S^*$.

Theorem 3.1. *Assume $S^0 \in (0, S^*)$. Then for any $c \geqslant -v$, there is no nonnegative traveling wave solutions $(S(x,t), u(x,t)) = (\phi_c(x+ct), \psi_c(x+ct))$ satisfying*

$$\phi_c(-\infty) = S^0, \qquad \phi_c(+\infty) = S_0 < S^0, \qquad \psi_c(\pm\infty) = 0, \qquad \psi_c(\cdot) \not\equiv 0. \tag{3.1}$$

Proof. We prove the theorem by a contradiction argument. Assume that there exists a nonnegative traveling wave solution $(S(x,t), u(x,t)) = (\phi_c(x+ct), \psi_c(x+ct))$ satisfying (3.1). Similar to Theorem 2.6, it is easy to see that $\phi'_c(\xi) \leqslant 0$ and hence $\phi_c(\xi) < S^*$ and $\int_{-\infty}^{\infty} \psi_c(\xi)\,d\xi < \infty$. Note that $\psi_c(\xi)$ satisfies

$$-\psi''_c(\xi) + (c+v)\psi'_c(\xi) + k\psi_c(\xi) = e^{-d\tau}\int_{-\infty}^{+\infty} f\big(\phi_c\big(\xi-y-(c+v)\tau\big)\big)\psi_c\big(\xi-y-(c+v)\tau\big)f_\alpha(y)\,dy.$$

Let

$$\Lambda'_1 = \frac{(c+v) - \sqrt{(c+v)^2 + 4k}}{2}, \qquad \Lambda'_2 = \frac{(c+v) + \sqrt{(c+v)^2 + 4k}}{2}.$$

Then one has

$$\psi_c(\xi) = \frac{1}{\Lambda'_2 - \Lambda'_1}\left[\int_0^{\infty} e^{\Lambda'_1\eta}H(\xi-\eta)\,d\eta + \int_{-\infty}^{0} e^{\Lambda'_2\eta}H(\xi-\eta)\,d\eta\right],$$

where

$$H(x) = e^{-d\tau}\int_{-\infty}^{+\infty} f\big(\phi_c\big(x-y-(c+v)\tau\big)\big)\psi_c\big(x-y-(c+v)\tau\big)f_\alpha(y)\,dy.$$

Since

$$\begin{aligned}\int_{-\infty}^{\infty} H(x)\,dx &< e^{-d\tau} f\big(S^*\big)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\psi_c\big(x-y-(c+\nu)\tau\big)f_\alpha(y)\,dy\,dx\\ &= e^{-d\tau} f\big(S^*\big)\int_{-\infty}^{+\infty}\psi_c(x)\,dx\\ &= k\int_{-\infty}^{+\infty}\psi_c(x)\,dx,\end{aligned}$$

we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\psi_c(x)\,dx &= \frac{1}{\Lambda_2'-\Lambda_1'}\left[\int_{-\infty}^{+\infty}\int_0^{\infty} e^{\Lambda_1'\eta}H(\xi-\eta)\,d\eta\,d\xi + \int_{-\infty}^{+\infty}\int_{-\infty}^{0} e^{\Lambda_2'\eta}H(\xi-\eta)\,d\eta\,d\xi\right]\\ &< \frac{k}{\Lambda_2'-\Lambda_1'}\left[\int_0^{\infty} e^{\Lambda_1'\eta}\,d\eta + \int_{-\infty}^{0} e^{\Lambda_2'\eta}\,d\eta\right]\int_{-\infty}^{+\infty}\psi_c(x)\,dx\\ &= \frac{k}{\Lambda_2'-\Lambda_1'}\left(\frac{1}{\Lambda_2'}-\frac{1}{\Lambda_1'}\right)\int_{-\infty}^{+\infty}\psi_c(x)\,dx\\ &= \int_{-\infty}^{+\infty}\psi_c(x)\,dx,\end{aligned}$$

which is a contradiction. This completes the proof. □

4. Discussion

In this paper, we have studied the existence and non-existence of nontrivial traveling wave solutions for the model (1.2). Theorems 2.7 and 3.1 combined provide a threshold condition for the existence of traveling wave solutions, which is valid for arbitrary $d_1 \geqslant 0$ and $\tau > 0$. Here we conjecture that the linear speed $c_* > 0$ corresponds to the minimal wave speed of nontrivial traveling wave solutions. We can give a proof as in [19] but here the details are omitted.

In addition, it is not difficult to see that the results of Theorems 2.7 and 3.1 remain valid for the following two systems:

$$\begin{cases}\dfrac{\partial}{\partial t}S(x,t) = d_1\Delta S(x,t) - f\big(S(x,t)\big)I(x,t),\\ \dfrac{\partial}{\partial t}I(x,t) = \Delta I(x,t) + \displaystyle\int_{-\infty}^{t}\int_{-\infty}^{+\infty} f\big(S(y,s)\big)I(y,s)K(x-y,t-s)\,dy\,ds - \gamma I(x,t),\end{cases}$$

and

$$\begin{cases}\dfrac{\partial}{\partial t}S(x,t) = d_1\Delta S(x,t) - f\big(S(x,t)\big)\displaystyle\int_{-\infty}^{t}\int_{-\infty}^{+\infty} I(y,s)K(x-y,t-s)\,dy\,ds,\\ \dfrac{\partial}{\partial t}I(x,t) = \Delta I(x,t) + f\big(S(x,t)\big)\displaystyle\int_{-\infty}^{t}\int_{-\infty}^{+\infty} I(y,s)K(x-y,t-s)\,dy\,ds - \gamma I(x,t),\end{cases} \tag{4.1}$$

where f satisfies (A1)–(A3) and the kernel K satisfies

(K1) K is nonnegative and integrable, and satisfies

$$\int_0^{\infty}\int_{-\infty}^{\infty} K(x,t)\,dx\,dt = 1 \quad\text{and}\quad K(x,t) = K(-x,t),\quad (x,t)\in\mathbb{R}\times[0,\infty);$$

(K2) For every $c \geqslant 0$, there exists $\lambda_c \in (0, +\infty]$ such that $\int_0^\infty \int_{-\infty}^\infty K(x,t)e^{-\lambda(x+ct)}\,dx\,dt < +\infty$ for any $\lambda \in [0, \lambda_c)$, and $\int_0^\infty \int_{-\infty}^\infty K(x,t)e^{-\lambda(x+ct)}\,dx\,dt \to +\infty$ as $\lambda \to \lambda_c - 0$.

In fact, when $f(S) = \beta S$, system (4.1) has been studied by Wang and Wu [19].

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